$F : \mathbb{R} \longrightarrow \mathbb{R}^+$

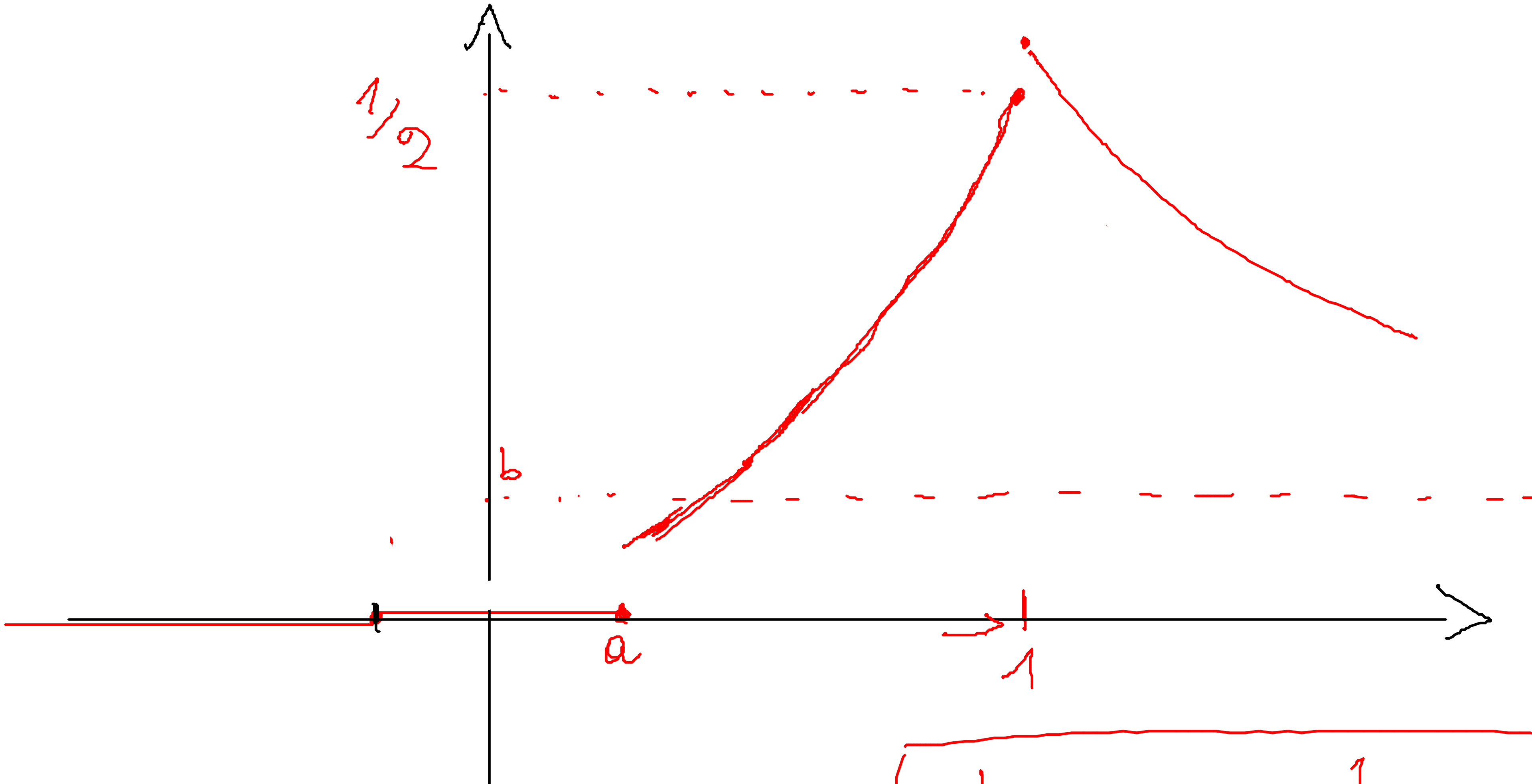
$$F(x) = \begin{cases} 0 & \text{per } x \le a \\ \frac{1}{2}x^2 & \text{per } a < x < 1 \\ b + c\,e^{-x} & \text{per } x \ge 1 \end{cases}$$

a, b, c be costanti, $a < 1$

(a) Det. a, b e c in modo che F sia una funz. di ripartizione

(b) Se $X \sim F$ (con a, b, c determinate in (a)) Calcolare $P(X = 1)$ e $P(X \ge 1)$

(c) Det a, b, c in modo che F sia continua



$$b + c\,e^{-x}$$

$$1 > a \geq 0$$

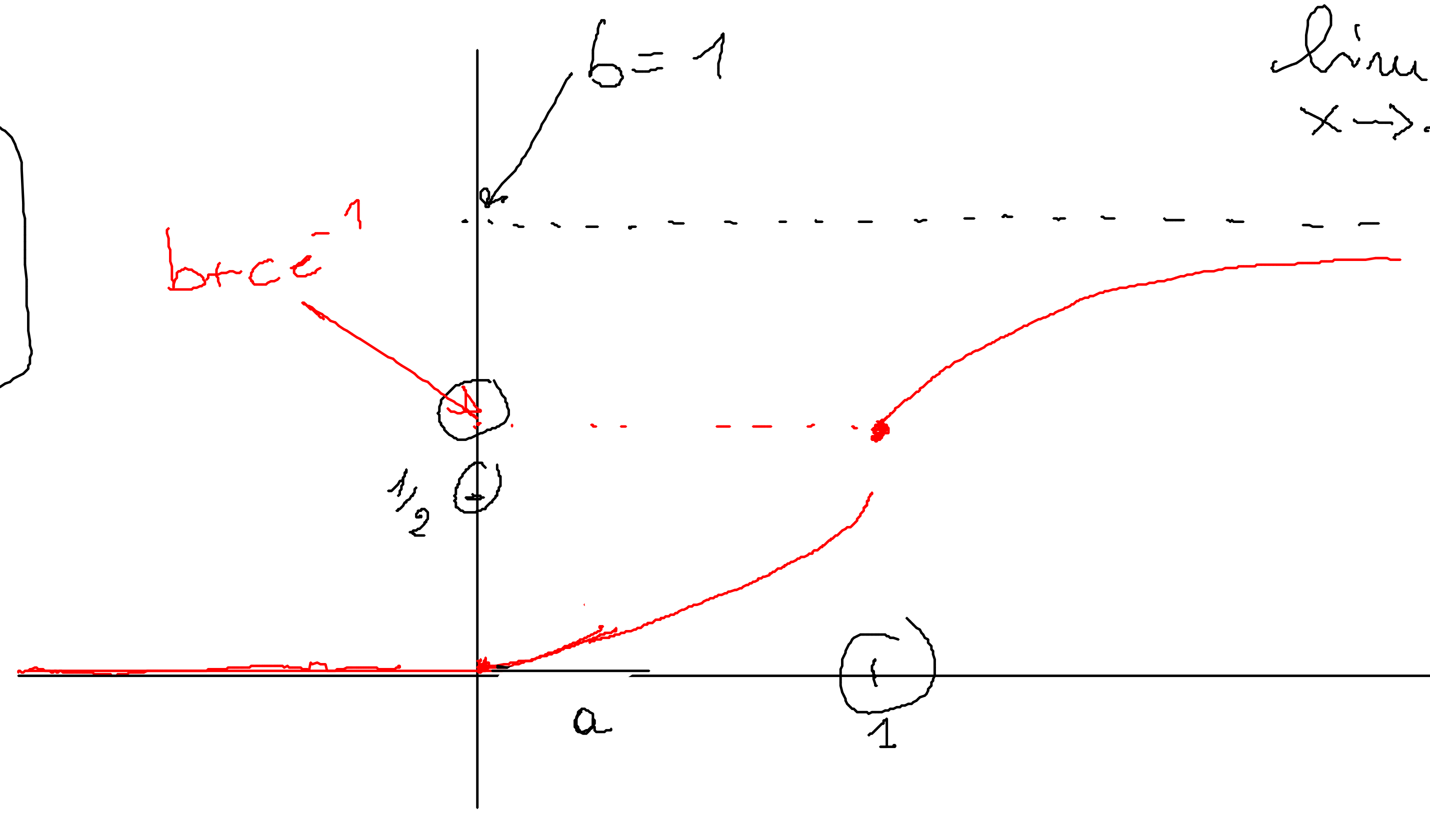
$$\boxed{b + c\,e^{-1} \geq \frac{1}{2}}$$

$a = 0$

$b = 1$

$-\frac{e}{2} \le c \le 0$

$$\lim_{x \to +\infty} b + ce^{-x} = b$$



$\cancel{0 \le a < 1}$

$a = 0$

$b + ce^{-1} \ge \frac{1}{2}$

$c \le 0$

$1 + ce^{-1} \ge \frac{1}{2}$

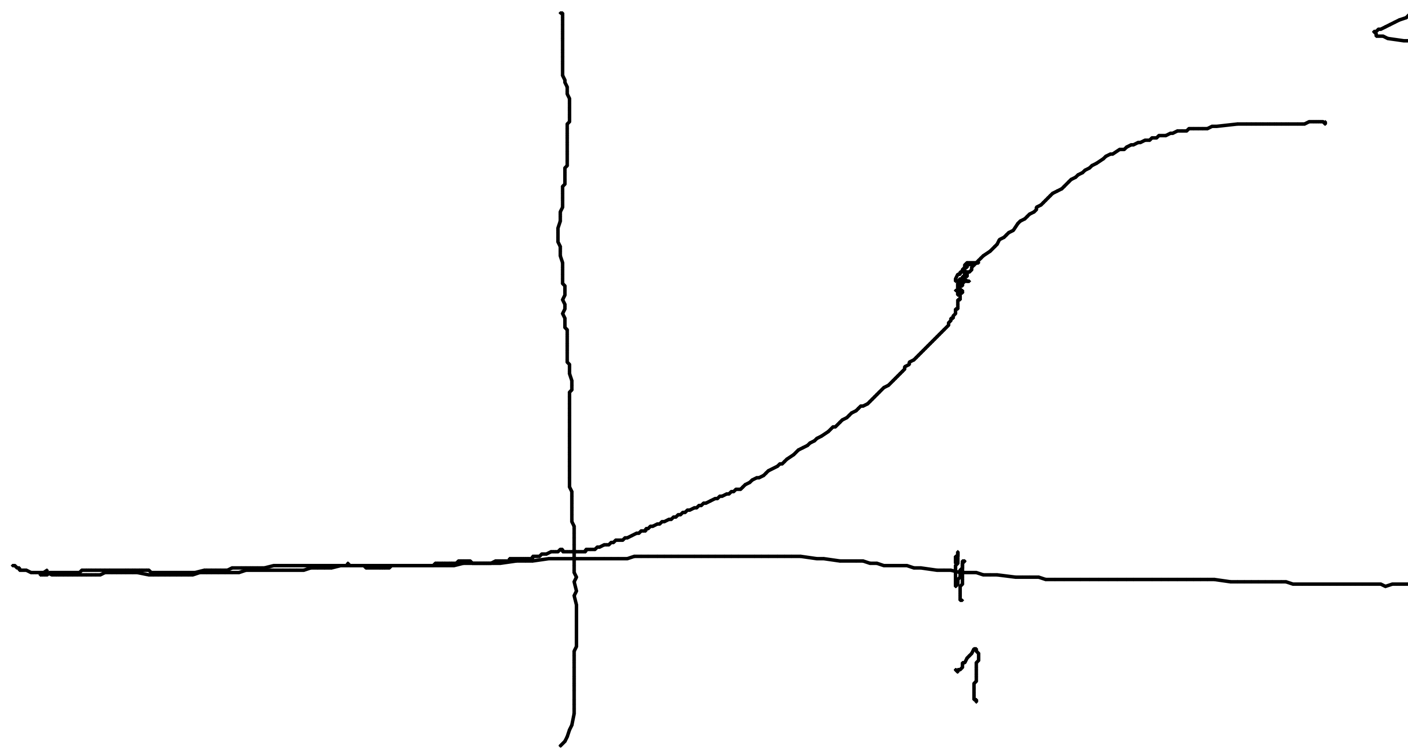
$ce^{-1} \ge -\frac{1}{2}$

$b = 1$

$\boxed{c \ge -\frac{e}{2}}$

(c) $$\frac{1}{2} = 1 + c e^{-1}$$

$$c = -\frac{e}{2}$$



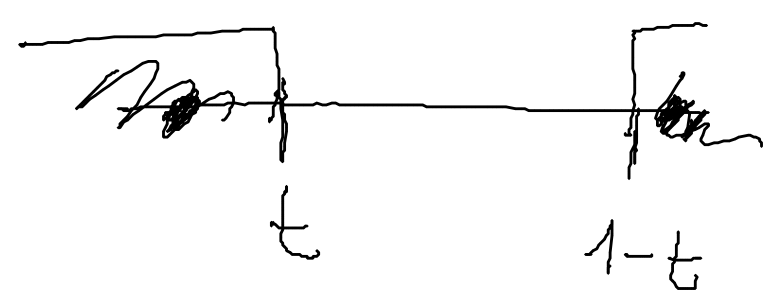
Sia $X \sim \mathcal{U}([0,1])$

(a) Calc. la legge di

$$Y = \max(X, 1-X)$$

(b) Calc. $P(Y \leq 3 \mid X \leq \frac{1}{2})$

(c) Calc. Cov (X, Y)

(a) $P(Y \leq t) =$ 

$$= P(\max(X, 1-X) \leq t) =$$

$$= P(\{X \leq t\} \cap \{1-X \leq t\})$$

$$= P(X \leq t, X \geq 1-t)$$

$$= \begin{cases} 0 & t < 1-t \rightarrow t < \frac{1}{2} \\ P(1-t \leq X \leq t) & 1-t \leq t \rightarrow t \geq \frac{1}{2} \end{cases}$$

$$X \sim F \qquad F(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$$

$$P(a \le X \le b) =$$

$$= F(b) - \lim_{t \to a^-} F(t) =$$

$$= F(b) - F(a)$$

$$P(1 - t \le X \le t) = F(t) - F(1-t)$$

$$t \ge \frac{1}{2}$$

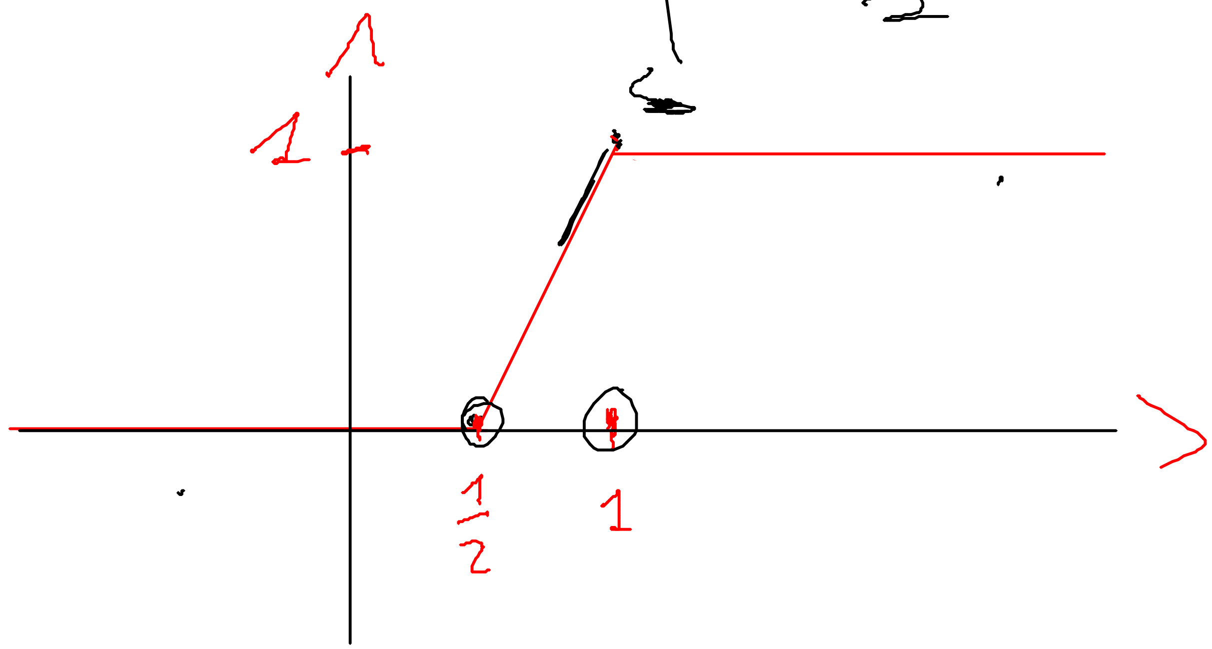
$$F(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$$

$$F(1-t) = \begin{cases} 0 & 1-t < 0 \rightarrow \quad t > 1 \\ 1-t & 0 \le 1-t \le 1 \qquad 0 \le t \le 1 \\ 1 & 1-t > 1 \qquad t < 0 \end{cases}$$

$$F(t) - F(1-t) = \begin{cases} t-(1-t) & \frac{1}{2} \le t \le 1 \\ 1-0 & t > 1 \end{cases} = \begin{cases} 2t-1 \\ 1 \end{cases}$$

$$P(Y \le t) = \begin{cases} 0 & t < \frac{1}{2} \\ 2t - 1 & \frac{1}{2} \le t \le 1 \\ 1 & t > 1 \end{cases}$$

$$\sim \mathcal{U}\left(\left[\tfrac{1}{2}, 1\right]\right)$$



$$P(Y \le t) = \begin{cases} 0 & t < a \\ \dfrac{t-a}{b-a} & a \le t \le b \\ 1 & t > b \end{cases}$$

$a = \frac{1}{2}$

$b = 1$

$$\frac{t - \frac{1}{2}}{1 - \frac{1}{2}} = 2t - 1$$

$$\begin{cases} 0 & t < \frac{1}{2} \\ 2t-1 & \frac{1}{2} \le t \le 1 \\ 1 & t > 1 \end{cases}$$

$$P(Y \leq t) = \begin{cases} 0 & t < \frac{1}{2} \\ 2t - 1 & \frac{1}{2} \leq t \leq 1 \\ 1 & t > 1 \end{cases}$$

$$g(t) = \begin{cases} 0 & t \leq \frac{1}{2} \\ 2 & \frac{1}{2} < t < 1 \\ 0 & t \geq 1 \end{cases}$$

$$h(t) = \begin{cases} \dfrac{1}{b-a} & a \le t \le b \\ 0 & \end{cases}$$

$$a = \frac{1}{2} \qquad b = 1$$

$$\frac{1}{b-a} = \frac{1}{1-\frac{1}{2}} = 2$$

$$X \sim F$$

$$P(X=x) = F(x) - \lim_{t \to x^-} F(t)$$

$$P(X \geq 1) = 1 - \lim_{t \to x^-} F(t)$$

$$P(X=1) =$$

$$= F(1) - \lim_{t \to 1^-} F(t) =$$

$$= \left(1 + ce^{-1}\right) - \frac{1}{2} = \frac{1}{2} + ce^{-1}$$

$$P(X \geq 1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$Y \sim \mathcal{U}\left(\left[\frac{1}{2}, 1\right]\right)$$

$$P\left(Y \leq \frac{3}{4} \,\middle|\, X \leq \frac{1}{2}\right)$$

$$Y = \max(X, 1-X)$$

$$\boxed{0 \leq X \leq 1}$$

$$f_X(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \end{cases}$$

$$0 \leq 1 - X \leq 1$$

$$\boxed{0 \le Y \le 1}$$

$$\begin{cases} 0 \\ ? \\ 0 \end{cases}$$

$t < 0$

$t > 1$

$$\boxed{\frac{P(A|B)}{P(A\cap B)}}\quad P\left(Y \le \frac{3}{4} \,\middle|\, X \le \frac{1}{2}\right) =$$

$$= \frac{P\left(Y \le \frac{3}{4},\ X \le \frac{1}{2}\right)}{P\left(X \le \frac{1}{2}\right)} =$$

$$\frac{P\left(\frac{1}{4} \le X \le \frac{1}{2}\right)}{P\left(X \le \frac{1}{2}\right)} = \frac{P\left(1 - X \le \frac{3}{4},\ X \le \frac{1}{2}\right)}{P\left(X \le \frac{1}{2}\right)} =$$

$$= \frac{P\left(\frac{1}{4} \leq X \leq \frac{1}{2}\right)}{P\left(X \leq \frac{1}{2}\right)} = \frac{F\left(\frac{1}{2}\right) - F\left(\frac{1}{4}\right)}{F\left(\frac{1}{2}\right)} =$$

$$= \frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

X e Y sono indipendenti?

$$Y = \varphi(X)$$

$$P(X\in A, Y\in B) = P(X\in A)\cdot P(Y\in B) \quad \forall A, \forall B$$

Non vale

Trovare A e B in modo che $P(X\in A, Y\in B) \neq$

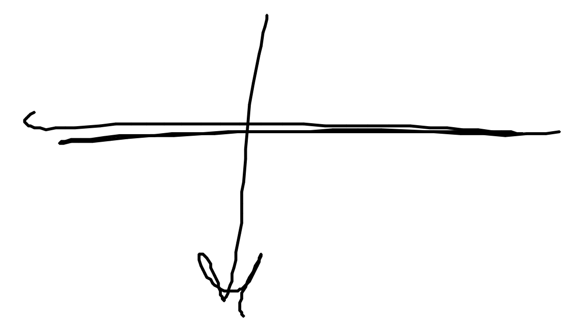
$$\neq P(X\in A)\, P(Y\in B)$$

$$\underbrace{P(Y \in B \mid X \in A)} \neq P(Y \in B)$$

$$\frac{1}{2} = P\left(Y \le \frac{3}{4} \mid X \le \frac{1}{2}\right) \neq P\left(Y \le \frac{3}{4}\right)$$

$$A = \left(-\infty, \frac{1}{2}\right]$$

$$B = \left(-\infty, \frac{3}{4}\right]$$



$$2t - 1$$

$$2 \cdot \frac{3}{4} - 1 = \frac{1}{2}$$

$$= P(\underbrace{X \in A}, \underbrace{Y \in B}) \neq \underbrace{P(X \in A)}\,\underbrace{P(Y \in B)}$$

$$P\left(X \le \frac{1}{8}\right) = \frac{1}{8}$$

$$P\left(Y \le \frac{3}{4}\right) = \frac{1}{2}$$

$$P\Big(\underbrace{X \le \frac{1}{8}}_{X \le \frac{1}{8}},\ \underbrace{Y \le \frac{3}{4}}_{1 - X \le \frac{3}{4}}\Big)$$

$$1 - X \le \frac{3}{4} \;\Rightarrow\; X \ge \frac{1}{4}$$

$X \sim f$

$Y \sim g$

$$\int_{-\infty}^{u} dx \int_{-\infty}^{v} dy\, f(x)\, g(y)$$

$$h(x, y) = f(x) \cdot g(y)$$

$$P(X \leq u, Y \leq v) = \int_{-\infty}^{u} dx \int_{-\infty}^{v} dy\; h(x, y)$$

$$= \left(\int_{-\infty}^{u} f(x)\,dx\right)\left(\int_{-\infty}^{v} g(y)\,dy\right)$$

$$= \underbrace{P(X \leq u)} \cdot \underbrace{P(Y \leq v)}$$

$$= P(X \leq u, Y \leq v)$$

$$p(x,y) = p_X(x)\, p_Y(y)$$

$$P(X \in A, Y \in B) =$$

$$= \sum_{(x,y) \in A \times B} p(x,y) = \sum_{\substack{x \in A \\ y \in B}} p_X(x)\, p_Y(y)$$

$$= \left(\sum_{x \in A} p_X(x) \right) \left(\sum_{y \in B} p_Y(y) \right) = P(X \in A) \cdot P(Y \in B)$$

$$\operatorname{Cov}(X, Y) = 0 = \frac{3}{8}$$

$$= E[XY] - E[X]E[Y]$$

$$E[X] = \frac{1}{2}$$

$$E[Y] = \frac{3}{4}$$

$$\frac{3}{8} - \frac{3}{8} = 0$$

$$E[XY] = E[\varphi(X,Y)]$$

$p(x,y)$

$\varphi(X,Y) = xy$

$$= \sum_{x,y} x\, y\, p(x,y)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, y\, h(x,y)\, dx\, dy =$$

$$\left(\int_{-\infty}^{+\infty} x\, f(x)\, dx\right)\left(\int_{-\infty}^{+\infty} y\, g(y)\, dy\right) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} x\, y\, h(x,y)\, dy$$

$E[X]$ $E[Y]$ $f(x)$

$$E[XY] = E[\underbrace{X\varphi(X)}] =$$

$$= \int_{-\infty}^{+\infty} \underbrace{x \max(x, 1-x)} \, \underline{f(x)} \, dx =$$

densité unif. su $[0, 1]$ a b

$$\begin{cases} \frac{1}{b-a} \\ 0 \end{cases}$$

$$= \cancel{\int_{-\infty}^{0}} + \int_{0}^{1} + \cancel{\int_{1}^{+\infty}} =$$

$$= \int_0^1 x \max(x, 1-x) \, 1 \, dx$$

$$\int_0^1 x \max(x, 1-x)\, dx =$$

$$\frac{3-1+8-1}{24} = \frac{9}{24}$$

$$= \int_0^{1/2} x(1-x)\, dx + \int_{\frac{1}{2}}^{1} x^2\, dx$$

$$= \int_0^{1/2} (x - x^2)\, dx + \int_{1/2}^{1} x^2\, dx = \cancel{\frac{1}{8} - 1} \; \frac{3}{8}$$

$$\left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^{1/2} + \left.\frac{x^3}{3}\right|_{1/2}^{1} = \frac{1}{8} - \frac{1}{24} + \frac{1}{3} - \frac{1}{24} \;//$$